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Note

Bonferroni-type inequalities and binomially bounded functions

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ABSTRACT

We present a unified approach to an important subclass of Bonferroni-type inequalities by considering the so-called binomially bounded functions. Our main result associates with each binomially bounded function a Bonferroni-type inequality. By appropriately choosing this function, several well-known and new results are deduced in a concise and unified way.

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1. Introduction

Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathscr{A}, P)$. The classical Bonferroni inequalities state that for any $r \in \mathbb{N}$,

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) \tag{1}$$

where $\mathbb{N} := \{1, 2, \ldots\}$. There are a lot of improvements and applications of these inequalities; see e.g., [3] for a detailed survey and [1] for some recent developments.

Variants of the classical Bonferroni inequalities, which are valid for any finite family of events, are often referred to as *Bonferroni-type inequalities*. A well-known Bonferroni-type inequality is the following improvement of (1) due to Galambos [2]:

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \frac{r+1}{|V|} \sum_{\substack{I\subseteq V\\ |I|=r+1}} P\left(\bigcap_{i\in I} A_i\right). \tag{2}$$

One of the most notable contributions in this field is due to Grable [4]. Following Grable [4], a k-uniform hypergraph $H = (V, \mathscr{E})$ is called *sparse* if for any non-empty subset W of V the induced subhypergraph $H[W] := (W, \mathscr{E} \cap 2^W)$ has at most $\binom{|W|-1}{k-1}$ hyperedges. Grable's result states that the classical Bonferroni inequalities (1) can be improved by adding terms corresponding to the hyperedges of any sparse $(r+1)$-uniform hypergraph $H = (V, \mathscr{E})$:

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{I\in\mathscr{E}} P\left(\bigcap_{i\in I} A_i\right). \tag{3}$$

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Grable [4] also identifies a subclass of all sparse $(r+1)$-uniform hypergraphs, the so-called k-matroid trees, for which the greedy algorithm finds an optimal solution. Particular cases of Grable's inequality (3) are Hunter's inequality [6], which is obtained if H is a tree, and Tomescu's inequality [7], which is obtained if H is a so-called hypertree.

In this paper, we establish a generalization of (2) and (3) by introducing the novel concept of binomially bounded functions. In this way, several well-known and new results are obtained in a concise and unified way.

Throughout this paper, the results remain valid if P is replaced by some arbitrary finite measure (e.g., the counting measure) on the algebra generated by the sets A_v, $v \in V$.

2. Binomially bounded functions

The concept of a binomially bounded function arose from the proof of our main result and its consequences in Section 3.

Definition 1. For any finite set V and any $k \in \mathbb{N}$ we use $[V]^k$ to denote the set of k-subsets of V. A function $f : [V]^k \to \mathbb{R}$ is called *binomially bounded* if

$$\sum_{\substack{I\subseteq W\\ |I|=k}} f(I) \le \binom{|W|-1}{k-1}$$

for any non-empty subset W of V.

Remark. As a consequence of the preceding definition we observe that any binomially bounded function $f : [V]^k \to \mathbb{R}$ satisfies $f(I) \le 1$ for any $I \in [V]^k$.

Examples of binomially bounded functions are provided by the following propositions.

Proposition 2. *Let V be a finite set and p_v, $v \in V$, be non-negative reals such that $\sum_{v\in V} p_v \le 1$. Then, for any $k \in \mathbb{N}$ the function*

$$f : [V]^k \to \mathbb{R}, \qquad f(I) := \sum_{i\in I} p_i$$

is binomially bounded.

Proof. For any non-empty subset W of V we find that

$$\sum_{\substack{I\subseteq W\\ |I|=k}} f(I) = \sum_{\substack{I\subseteq W\\ |I|=k}} \sum_{i\in I} p_i = \sum_{i\in W} p_i \sum_{\substack{I\subseteq W\\ |I|=k\\ i\in I}} 1 = \sum_{i\in W} p_i \binom{|W|-1}{k-1} \le \binom{|W|-1}{k-1},$$

which proves the statement. $\square$

Proposition 3. *For any finite set V and any $k \in \mathbb{N}$ the function $f : [V]^k \to \mathbb{R}$ which is defined by $f(I) := |I|/|V|$ for any $I \in [V]^k$ is binomially bounded.*

Proof. Define $p_v := 1/|V|$ for any $v \in V$, and apply Proposition 2. $\square$

Proposition 4. *Let $H = (V, \mathscr{E})$ be a sparse k-uniform hypergraph. Then, the function $f : [V]^k \to \{0, 1\}$, which is defined by $f(I) := 1$ if and only if $I \in \mathscr{E}$, is binomially bounded. In particular, the edge indicator function of any tree is binomially bounded.*

Proof. For any non-empty subset W of V, the left-hand side of (1) gives the number of hyperedges of $H[W]$. Since H is sparse, the number of hyperedges of $H[W]$ is at most $\binom{|W|-1}{k-1}$. $\square$

Remark. Note that any binomially bounded function $f : [V]^k \to \{0, 1\}$ gives rise to a sparse k-uniform hypergraph $H = (V, \{I \in [V]^k | f(I) = 1\})$. In particular, any binomially bounded function $f : [V]^2 \to \{0, 1\}$ gives rise to a forest. In view of this and the preceding proposition, the concept of a sparse k-uniform hypergraph turns out to be equivalent to that of a 0,1-valued binomially bounded function.

3. Main result and consequences

We are now ready to state our main result.

Theorem 5. *Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathscr{A}, P)$. Then, for any $r \in \mathbb{N}$ and any binomially bounded function $f : [V]^{r+1} \to \mathbb{R}$ we have*

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \ge (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\subseteq V\\ |I|=r+1}} f(I) P\left(\bigcap_{i\in I} A_i\right).$$

Proof. By the method of indicators [3,5] it suffices to prove that

$$(-1)^r 1_{\bigcup_{v\in V} A_v} \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} 1_{\bigcap_{i\in I} A_i} + \sum_{\substack{I\subseteq V\\ |I|=r+1}} f(I) 1_{\bigcap_{i\in I} A_i} \tag{4}$$

where 1_A denotes the indicator function of A, that is, $1_A(\omega) = 1$ if $\omega \in A$, and $1_A(\omega) = 0$ if $\omega \notin A$. In order to prove (4) we show that for any $\omega \in \bigcup_{v\in V} A_v$,

$$(-1)^r 1_{\bigcup_{v\in V} A_v}(\omega) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} 1_{\bigcap_{i\in I} A_i}(\omega) + \sum_{\substack{I\subseteq V\\ |I|=r+1}} f(I) 1_{\bigcap_{i\in I} A_i}(\omega). \tag{5}$$

Now, $1_{\bigcup_{v\in V} A_v}(\omega) = 1$ for any $w \in \bigcup_{v\in V} A_v$, while $1_{\bigcap_{i\in I} A_i}(\omega) = 1$ if and only if $\omega \in A_i$ for any $i \in I$, or equivalently, if $I \subseteq \{i \in V \mid \omega \in A_i\} =: V_\omega$; otherwise $1_{\bigcap_{i\in I} A_i}(\omega) = 0$. Leaving out these vanishing terms, our claim (5) becomes

$$(-1)^r \geq (-1)^r \sum_{\substack{I\subseteq V_\omega\\ 0<|I|\leq r}} (-1)^{|I|-1} + \sum_{\substack{I\subseteq V_\omega\\ |I|=r+1}} f(I). \tag{6}$$

Since f is binomially bounded, we find that

$$\sum_{\substack{I\subseteq V_\omega\\ |I|=r+1}} f(I) \leq \binom{|V_\omega|-1}{r} = (-1)^r \sum_{\substack{I\subseteq V_\omega\\ |I|\leq r}} (-1)^{|I|} = (-1)^r - (-1)^r \sum_{\substack{I\subseteq V_\omega\\ 0<|I|\leq r}} (-1)^{|I|-1},$$

where the first equals sign comes from the well-known combinatorial identity

$$\sum_{k=0}^{r} (-1)^k \binom{m}{k} = (-1)^r \binom{m-1}{r} \quad (m \in \mathbb{N}, r \in \mathbb{N}).$$

Thus (6) is established, and the proof of the theorem is complete. □

Remark. By a suitable choice of f several known and new results can be deduced. For instance, by choosing f according to Propositions 3 and 4 we obtain Galambos' inequality (2) resp. Grable's inequality (3).

The next inequality, which is new even for $r = 1$, is a consequence of Theorem 5 and Proposition 2. As pointed out by one of the referees, this new inequality can also be deduced by averaging Grable's inequality.

Corollary 6. *Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathscr{A}, P)$, and p_v, $v \in V$, be non-negative reals such that $\sum_{v\in V} p_v \leq 1$. Then, for any $r \in \mathbb{N}$,*

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\subseteq V\\ |I|=r+1}} P\left(\bigcap_{i\in I} A_i\right) \sum_{i\in I} p_i.$$

The following inequality, also new for $r = 1$, is a specialization of the preceding one. It agrees with (2) if all probabilities $P(A_v)$ are equal for all $v \in V$, or if all probabilities $P(\bigcap_{i\in I} A_i)$ are equal for all subsets $I \subseteq V$ satisfying $|I| = r + 1$.

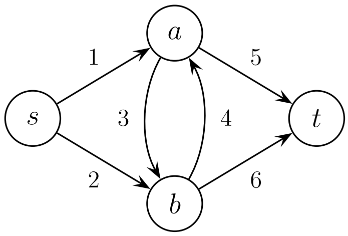
Corollary 7. *Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathscr{A}, P)$ such that $P(A_v) > 0$ for at least one $v \in V$. Then, for any $r \in \mathbb{N}$,*

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\subseteq V\\ |I|=r+1}} P\left(\bigcap_{i\in I} A_i\right) \sum_{i\in I} P(A_i) \Big/ \sum_{v\in V} P(A_v).$$

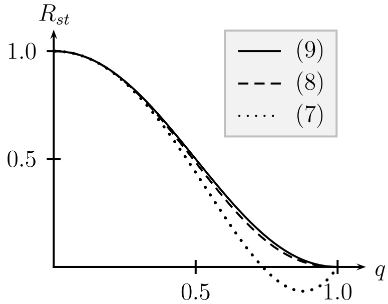
4. Comparison with Grable's bound

Consider the bridge network in Fig. 1(a) whose nodes are perfectly reliable and whose edges fail randomly and independently with probability q ($0 \leq q \leq 1$). Let R_{st} denote the source-to-terminal reliability of this network, that is, the probability that a message can be sent from s to t along a path of operating edges. In order to obtain a lower bound on this reliability, let A_1 denote the event that both edges 1 and 2 fail, A_2 the event that edges 1, 4 and 6 fail, A_3 the event that edges 2, 3 and 5 fail, and A_4 the event that edges 5 and 6 fail. Then, $P(A_1) = P(A_4) = q^2$, $P(A_2) = P(A_3) = q^3$, $P(A_2 \cap A_3) = q^6$, and $P(A_i \cap A_j) = q^4$ for all other choices of distinct i and j. Then, we have

$$1 - R_{\mathrm{st}} = P(A_1 \cup A_2 \cup A_3 \cup A_4).$$



(a) The bridge network.



(b) Bounds and exact reliability.

Fig. 1. Comparison with Grable's bound.

For $r = 1$ the best possible Grable bound (= Hunter's bound in this case) is given by

$$1 - R_{st} \leq P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4),$$

which simplifies to

$$R_{st} \geq 1 - 2q^2 - 2q^3 + 3q^4. \tag{7}$$

The binomially bounded function $f : [\{1, 2, 3, 4\}]^2 \to \mathbb{R}$, which is defined by

$$f(\{1, 2\}) = f(\{1, 3\}) = f(\{1, 4\}) = f(\{2, 4\}) = 1,$$
$$f(\{2, 3\}) = -1, \qquad f(\{3, 4\}) = 0,$$

leads via Theorem 5 to the estimate

$$1 - R_{st} \leq P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_4) + P(A_2 \cap A_3),$$

which simplifies to

$$R_{st} \geq 1 - 2q^2 - 2q^3 + 4q^4 - q^6. \tag{8}$$

Note that (8) is uniformly (that is, for all q in the interval $[0, 1]$) better than (7). Fig. 1(b) shows the bounds (7) and (8) as well as the exact reliability

$$R_{st} = 1 - 2q^2 - 2q^3 + 5q^4 - 2q^5. \tag{9}$$

From this example we conclude that Theorem 5 not only generalizes Grable's bound (cf. Section 3), but also improves it. The particular choice of f in this example was proposed in a nearby fashion by an anonymous referee whom we would like to thank very much for this hint.

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